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LETTER TO THE EDITOR

Mean field renormalisation group for the disordered transverse Ising model

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Abstract. The mean field renormalisation group approach is applied to the bond diluted transverse Ising model. The critical surface in the temperature-transverse field–concentration space is obtained for the two- and three-dimensional models and estimates of critical exponents are also presented.

The mean field renormalisation group method (MFRG) has been recently proposed by Indekeu *et al* (1982) for computing critical properties of lattice spins systems. This method combines mean field results for small clusters of spins with renormalisation group ideas. It is based upon a comparison of the behaviour of clusters of different size in the presence of symmetry breaking boundary conditions (mean field) which simulate the effect of surrounding spins in the infinite system. While mean field theory identifies the order parameter of this field with the order parameter of the cluster, the MFRG assumes that the order parameter scales in the same way (Indekeu *et al* 1982).

The MFRG approach has been applied to classical (Ising and q-state Potts models) and quantum (transverse Ising model at zero temperature) pure spin systems (Indekeu *et al* 1982) and classical random systems including spin glass (Droz *et al* 1982). Quite good results have been obtained using just the simplest choice for the clusters namely, one- and two-spin clusters respectively. In several cases, the critical coupling is the same as the one obtained by the Bethe approximation. The critical exponents, though not very precise, are substantially better than those of mean field and Bethe calculations. More recently, the MFRG has also been applied to geometric phase transitions (De'Bell 1983) and the triangular Ising antiferromagnet (Slotte 1983).

In this letter we study the transverse Ising model (TIM) with quenched bond dilution described by the Hamiltonian

$$H = -\sum_{(i,j)} J_{ij}\sigma_i^z\sigma_j^z - \sum_i \Omega_i\sigma_i^x, \tag{1}$$

where the $\boldsymbol{\sigma}$'s are Pauli spin matrices, $\Omega_i = \Omega$ is the transverse field, and the sums run over sites on a d-dimensional lattice. The nearest-neighbour exchange coupling J_{ij} are random variables with probability distribution

$$P(J_{ij}) = (1-p)\delta(J_{ij}) + p\delta(J_{ij} - J). \tag{2}$$

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As conjectured by Harris (1974) this model should display a discontinuity in the critical transverse field when the bond concentration p reaches the critical value p_c at zero temperature. Earlier theoretical treatments of the diluted TIM such as series expansions (Elliott and Saville 1974) and CPA calculations (Lage 1976) were unable to pick out the zero temperature behaviour. Real space renormalisation group calculations on the two-dimensional bond diluted TIM (Stinchcombe 1981, dos Santos 1981, 1982) have picked up this discontinuity due to the presence of two fixed points at the percolation threshold. More recently, Saxena (1983) and Plascak (1983) have studied the two- and three-dimensional bond- and site-diluted TIM by using a two-spin cluster approximation and a variational procedure for the free energy respectively. The complete phase diagrams in the temperature-transverse field–concentration space agree with Harris's conjecture.

In order to study the critical properties of the model (1), we follow the MFRG procedure of Indekeu *et al* (1982) and consider the simplest choice for the clusters, as illustrated in figure 1 for $d=3$ with $N'=1$, and $N=2$ spins respectively. The

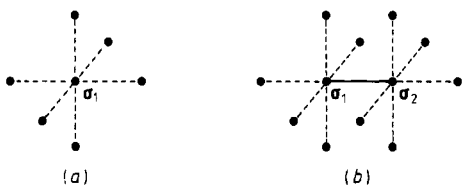

Figure 1. Schematic representation of one spin cluster (a) and two-spin cluster (b) in three dimensions. Broken lines represent interactions with nearest-neighbour spins.

z-component of the boundary spins in each cluster is fixed at b' and b respectively and, as in this case the phase transition is of second order, they are assumed to be very small. The Hamiltonian for the single spin then reads

$$H_1 = -\Omega'\sigma_1^x - \sum_{j=1}^{2d} J'_{1j}\sigma_1^z b', \tag{3}$$

where Ω' is the scaled transverse field and $\boldsymbol{\sigma}_i$ interacts with its $2d$ nearest neighbours through a term $-J'\sigma_i^z b'$. Diagonalisation of the Hamiltonian (3) is simple. After averaging over the disorder, the corresponding magnetisation per spin for small b' can be written as

$$m_1 = 2dp'b' \tanh(\alpha' K')/\alpha', \tag{4}$$

where $K' = J'/k_B T'$, $\alpha' = \Omega'/J'$ and p' is the scaled concentration of bonds. Similarly, the Hamiltonian of the two-spin cluster is given by

$$H_2 = -J_{12}\sigma_1^z\sigma_2^z - \Omega(\sigma_1^x + \sigma_2^x) - \sum_{j\neq 1} J_{1j}\sigma_1^z b - \sum_{j\neq 2} J_{2j}\sigma_2^z b. \tag{5}$$

In this case the spin $\boldsymbol{\sigma}_1$ interacts directly with $\boldsymbol{\sigma}_2$ through a term $-J\sigma_1^z\sigma_2^z$ and both $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ interact with their $(2d-1)$ nearest-neighbour spins through terms $-J\sigma_1^z b$ and $-J\sigma_2^z b$ respectively. Diagonalisation of H_2 is not simple. However, as we are interested in small values of the mean field, a perturbation expansion can be worked out in order to obtain the partition function in powers of b (Plascak 1983, Saxena

1983). The resulting magnetisation per spin, after averaging over the disorder, can be written as

$$m_2 = [p(1-p)(2d-1)\tanh(\alpha K)/\alpha + p^2(2d-1)S/\Delta]b, \tag{6}$$

where $K = J/k_B T$, $\alpha = \Omega/J$, and p is the bond concentration and

$$\Delta = 2\cosh K + 2\cosh KR,$$

$$S = e^{KR}C - e^{-KR}D - 2e^{K}/\alpha^2,$$

$$\begin{Bmatrix} C \\ D \end{Bmatrix} = [1/\alpha^2 + 2 \pm R/\alpha^2]/R,$$

$$R = (1+4\alpha^2)^{1/2}. \tag{7}$$

Imposing now a scaling relation of the form $m_1 = \xi m_2$ between such approximate magnetisations and assuming a similar scaling relation between the parameters b' and b (Indekeu *et al* 1982), one gets

$$2dp' \tanh(\alpha' K')/\alpha' = p(1-p)(2d-1)\tanh(\alpha K)/\alpha + p^2(2d-1)S/\Delta, \tag{8}$$

which is independent of ξ. Equation (8) can be viewed as a renormalisation recursion relation amongst K', α', p' and K, α, p. It is clear that one cannot determine the complete renormalisation flow diagram in the K, α, p space from this equation alone. It is, however, interesting to look first at the fixed point equation associated with equation (8):

$$(2dp - p + 1)\tanh(\alpha K)/\alpha = p(2d-1)S/\Delta. \tag{9}$$

Equation (9) is exactly the same as those obtained by Saxena (1983) and Plascak (1983) for the phase diagram of the disordered TIM. Thus, as far as the critical surface is concerned, the MFRG using the clusters illustrated in figure 1 is equivalent to the pair approximation used in previous works. Clearly, only a few points on this critical surface turn out to be real fixed points of a fully specified renormalisation transformation. Among them are the pure Ising model fixed point ($p = 1$, $\alpha = 0$) and the pure zero temperature TIM fixed point ($p = 1$, $K = \infty$). The logarithmic slope $(1/\alpha_c)\, d\alpha_c/dp = 0.91$ of the phase boundary at $p = 1$ and $T = 0$ for the two-dimensional model is comparable to the value 1.07 obtained by a real space renormalisation group calculation (dos Santos 1981). For $d = 3$ the corresponding slope is 0.94.

Although with the present approach we are unable to determine the complete flow in the parameter space of the Hamiltonian (1), equation (8) can be used to estimate critical exponents associated with some invariant sets in the K, α, p space. This can be done by computing

$$[\partial\mu'/\partial\mu]_{FP} = l^{y_\mu}, \tag{10}$$

where μ can be K, α, or p, $l = (N/N')^{1/d}$ is the rescaling factor and the derivative is taken at the fixed point of the particular invariant set considered. Table 1 lists some values of the critical exponents associated with the corresponding invariant set in the K, α, p space obtained from equations (8)–(10) in comparison with exact (if available) or other approximate results for the two-dimensional model. Besides the good results for the critical points, it is interesting to note that also a reasonable estimate is obtained for y_p which is given by

$$l^{y_p/d} = 1 + 1/2d, \tag{11}$$

Table 1. Critical points and critical exponents obtained from equations (8)–(10) in comparison with exact (if available) or other approximated results for the two-dimensional model.

$d=2$					
$p=1$	K_c	y_T			
$\alpha=0$	0.346	0.60			
	0.609	1.49	[a]		
	0.441	1.00	exact		
$p=1$	α_c	y_α			
$K=\infty$	3.334	0.700			
	1.55	2.016	[a]		
	3.08	1.587	[b]		
$\alpha=0$	p_c	y_p			
$K=\infty$	1/3	0.644			
	0.618	1.224	[a]		
	0.5[c]	3/4[d]	exact		
$K=\infty$	p_c	α_c	y_p	y_α	
	1/3	1.277	0.644	0.754	
	0.618	0.873	1.224	1.689	[a]

[a] Real space renormalisation group (dos Santos 1982).
[b] Ground state perturbation expansions (Pfeuty and Elliott 1971).
[c] Duality arguments (Sykes and Essam 1964).
[d] den Nijs conjecture for corresponding Potts model (den Nijs 1979).

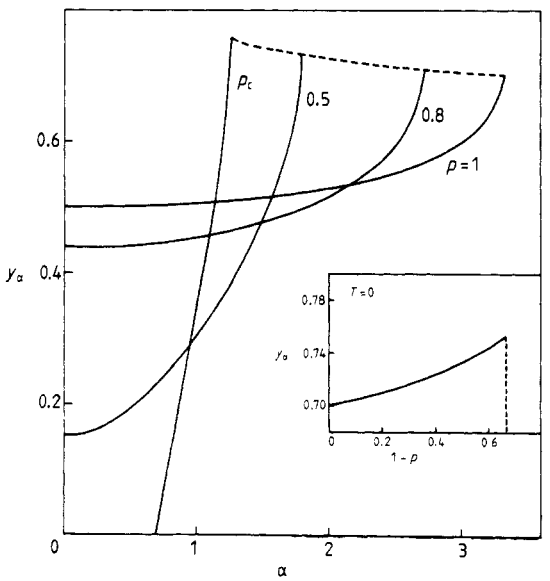

Figure 2. Critical exponent y_α as a function of α for various values of the concentration. The quantum ($T=0$) behaviour of y_α as a function of α is given by the broken line. The quantum behaviour as a function of the concentration is shown in the inset.

and is independent of any invariant set. Such independence does not, however, occur when computing either y_α or y_T. Figure 2 shows y_α as a function of α for different values of the concentration. The continuous variation with the transverse field, as well as some spurious negative values for $p \simeq p_c$, are a consequence of having only one recursion relation for the parameters of the system. However, a crossover between the classical system ($T \neq 0$) and the quantum system ($T = 0$) is apparent in this figure. The broken line shows the variation of y_α for the quantum system ($T = 0$) with the critical transverse field (or equivalently with the critical concentration shown in the inset). Another crossover driven by dilution is also apparent in this case.

Finally, the present approach is easily extended to dimensions above $d = 2$, where standard real space renormalisation methods are not practicable and other approximate results for the bond diluted TIM are definitely smaller. Table 2 shows some estimates of critical exponents obtained for the three-dimensional model. Similar curves to those illustrated in figure 2 can also be drawn for $d = 3$.

Table 2. The same as table 1 for the three-dimensional model.

$d = 3$				
$p = 1$	K_c	y_T		
$\alpha = 0$	0.203	0.651		
	0.214[a]	1.587[b]		
$p = 1$	α_c	y_α		
$K = \infty$	5.348	0.707		
	5.1	1.724[c]		
$\alpha = 0$	p_c	y_p		
$K = \infty$	0.2	0.667		
$K = \infty$	p_c	α_c	y_p	y_α
	0.2	1.277	0.667	0.784

[a] Domb (1974).
[b] Le Guillou and Zinn-Justin (1980).
[c] Pfeuty and Elliott (1971).

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